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NONEXISTENCE OF DECREASING EQUISINGULAR APPROXIMATIONS WITH LOGARITHMIC POLES

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Abstract. In this article, we present that for any complex manifold whose dimension is bigger than one, there exists a multiplier ideal sheaf such that there don't exist equisingular weights with logarithmic poles, which are not smaller than the orginal weight. A direct consequence is the nonexistence of decreasing equisingular approximations with logarithmic poles.

1. Introduction

Let φ be a plurisubharmonic function (see [8]) on a complex manifold X. Following Nadel [9], one can define the multiplier ideal sheaf $\mathcal{I}(\varphi)$ (with weight φ) to be the sheaf of germs of holomorphic functions f such that $|f|^2e^{-2\varphi}$ is locally integrable (see also [11], [12], [3], [4], etc.).

In [2] (see also [3]), Demailly shows that for any given quasi-plurisubharmonic function φ (i.e., locally can be expressed by $\psi + v$, where ψ is plurisubharmonic function and v is smooth) on compact Hermitian manifold M, there exist quasi-plurisubharmonic functions $\varphi_{S,j}$ $(j = 1, 2, \cdots)$ on M with smooth poles satisfying

$$\mathcal{I}(\varphi) = \mathcal{I}(\varphi_{S,j})$$

$(j = 1, 2, \cdots)$ ("equisingularity"), which are decreasing convergent to φ, when j goes to ∞.

It is called that a quasi-plurisubharmonic function φ_A has logarithmic poles if there exist holomorphic functions g_k $(k = 1, \cdots, N)$ such that

$$\varphi_A = c \log \sum_{k=1}^{N} |g_k|^2 + O(1),$$

where $c \in \mathbb{R}$ (see [2],[3]). In [2] (see also [3]), Demailly asked

Question 1.1. *For any given quasi-plurisubharmonic function φ on M, can one choose equisingular quasi-plurisuhbarmonic functions $\varphi_{A,j}$ $(j = 1, 2, \cdots)$ on M with logarithmic poles, which are decreasing convergent to φ $(j \to \infty)$?*

In this article, we give negative answers to Question 1.1 for any dimension $n \geq 2$ by the following theorem

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Theorem 1.2. *For any complex manifold M (compact or noncompact) $dimM \geq 2$ and $z_0 \in M$, there exists a quasi-plurisubharmonic function φ on M such that for any plurisubharmonic function $\varphi_A \geq \varphi$ near $z_0 \in M$ with logarithmic poles,*

$$c_{z_0}(\varphi) < c_{z_0}(\varphi_A) \tag{1.1}$$

holds, where $c_{z_0}(\varphi) := \sup\{c|\mathcal{I}(c\varphi)_{z_0} = \mathcal{O}_{z_0}\}$ is the complex singularity exponent of φ.

We prove Theorem 1.2 by considering the following

Remark 1.3. *Let*

$$\varphi_1 := \log(\max\{|z_1|, \cdots, |z_{n-1}|, |z_n|^a\}),$$

where $a \in (1, 1 + a/2)$ is a irrational number, and $(z_1, \cdots, z_n)$ are coordinates on $\mathbb{C}^n$. Let

$$\varphi_2 := \max\{\varphi_1 - 18n, 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n\}.$$

Let

$$\varphi := -M_\eta(-\varphi_2, 0),$$

where $M_\eta(t_1, t_2)$ is in Lemma (5.18) in [4]*, which satisfying*

(1) $M_\eta(t_1, t_2)$ is smooth on $\mathbb{R}^2$;

(2) $M_\eta(t_1, t_2)|_{\{t_2+2\varepsilon_0\leq t_1\}} = t_1$ and $M_\eta(t_1, t_2)|_{\{t_1+2\varepsilon_0\leq t_2\}} = t_2$,

and $\eta := (\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 = \frac{1}{1000}$.

In following two remarks present that φ in Remark 1.3 is quasi-plurisubharmonic, which can be can be extended to M.

Let

(1) $A_1 := \{z| \log(\max_{j=1,\cdots,n} |z_j|) < 0\}$;

(2) $A_2 := \{6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n < -2\varepsilon_0\}$;

(3) $A_3 := \{z| \log(\max_{j=1,\cdots,n} |z_j|) < -6\log n\}$.

It is clear that

$$A_3 \subset\subset A_1 \subset\subset A_2.$$

The following remark shows that φ in Remark 1.3 is quasi-plurisubharmonic.

Remark 1.4. *As*

$$\begin{aligned} 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n &\geq 12\log(\max_{j=1,\cdots,n} |z_j|) - 6n \\ &\geq a\log(\max_{j=1,\cdots,n} |z_j|) - 6n \geq \varphi_1 \end{aligned} \tag{1.2}$$

on A_1^c, then

$$\varphi_2(z)|_{A_1^c} = 6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n. \tag{1.3}$$

By (1) in Remark 1.3, it follows that φ is smooth on $(A_1^c)^o$.

As

$$\varphi_1|_{A_2} < 6\log(|z_1|^2 + \cdots + |z_n|^2) - 18n < 6n - 2\varepsilon_0 - 18n < -2\varepsilon_0$$

and

$$(6\log(|z_1|^2 + \cdots + |z_n|^2) - 6n)|_{A_2} < -2\varepsilon_0,$$

then it follows that $\varphi_2|_{A_2} < -2\varepsilon_0$. By using (2) in Remark 1.3, it follows that $\varphi|_{A_2} = \varphi_2$ is plurisubharmoic on A_2.

Note that

$$(A_1^c)^o \cup A_2 = \mathbb{C}^n.$$

Then φ in Remark 1.3 is quasi-plurisubharmonic.

The following remark shows that φ in Remark 1.3 can be extended to M.

Remark 1.5. *By equality 1.3 and and (2) in Remark 1.3, then it is clear that*

$$\begin{aligned}&\varphi|_{\{6\log(|z_1|^2+\cdots+|z_n|^2)-6n>2\varepsilon_0\}}\\&=-M_\eta(-6\log(|z_1|^2+\cdots+|z_n|^2)+6n,0)|_{\{6\log(|z_1|^2+\cdots+|z_n|^2)-6n>2\varepsilon_0\}}\equiv 0.\end{aligned}\tag{1.4}$$

The following remark present the singularity of φ in Remark 1.3

Remark 1.6. *As*

$$\begin{aligned}6\log(|z_1|^2+\cdots+|z_n|^2)-6n&\leq 12\log(\max_{j=1,\cdots,n}|z_j|)+6\log n-6n\\&\leq a\log(\max_{j=1,\cdots,n}|z_j|)-6n\leq\varphi_1\end{aligned}\tag{1.5}$$

on A_3, then

$$\varphi_2|_{A_3}=\varphi_1.$$

By Remark 1.4 ($\varphi|_{A_2}=\varphi_2$) and $A_3\subset\subset A_1$, it follows that

$$\varphi|_{A_3}=\varphi_1.$$

Using Theorem 1.2, we answer Question 1.1 by contradiction

Remark 1.7. *If not, then for the plurisubharmonic function $\varphi_1=\varphi|_{A_3}$ in Remark 1.3, there exists a plurisubharmonic function φ_A with logarithmic poles near o satisfying $c_o(\varphi_1)\varphi_A\geq c_o(\varphi_1)\varphi_1$, such that $e^{-2c_o(\varphi_1)\varphi_A}$ is not integrable near o. By Berndtsson's solution of the openness conjecture ([1]) posed by Demailly and Kollar ([7]), it follows that $c_o(\varphi_A)\leq c_o(\varphi_1)$, which contradicts Theorem 1.2.*

2. Some Preparations

In this section, we recall some known results and present some observations.

2.1. A sharp lower bound for the log canonical threshold for dimension 2 case.

In [6], Demailly and Hiep present the following

Theorem 2.1. *([6]) Let $\varphi_A\geq\varphi_1$ be a plurisubharmonic function near $o\in\mathbb{C}^2$ with logarithmic poles, then*

$$c_o(\varphi_A)\geq\frac{1}{e_1(\varphi_A)}+\cdots+\frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)},\tag{2.1}$$

where $e_k(\varphi_A):=\nu((dd^c\varphi_A)^k,o)$ ($e_1(\varphi_A)=\nu(\varphi_A,o)$).

As $\varphi_A\geq\varphi$, then one can obtain

$$e_n(\varphi_A)\leq e_n(\varphi)=a\tag{2.2}$$

and

$$e_k(\varphi_A)\leq e_k(\varphi)=1\ (k\in\{1,\cdots,n-1\})\tag{2.3}$$

(by using Second comparison theorem (7.8) and Example (6.11) in chapter III of [4])

2.2. **Observations.** Note that $c_o(\log\sum_{k=1}^N|g_k|^2)$ is a rational number (see [7]), and the Lelong number $\nu(\log\sum_{k=1}^N|g_k|^2,o)$ is a integer (see [4]), where g_k are holomorphic functions near $o\in\mathbb{C}^n$. Then it is clear that

Lemma 2.2. *Let plurisubharmonic function $\varphi_A:=c\log\sum_{k=1}^N|g_k|^2+O(1)$ near o, where $c\in\mathbb{R}^+$, and g_k are holomorphic functions near o. Then*

$$c_o(\varphi_A)\nu(\varphi_A,o)=c_o(\log\sum_{k=1}^N|g_k|^2)\nu(\log\sum_{k=1}^N|g_k|^2,o)$$

is a rational number.

We prove Theorem 1.2 by using the following lemma:

Lemma 2.3. *Let $\varphi_A\geq\varphi_1$ (as in Remark 1.3) be a plurisubharmonic function near $o\in\mathbb{C}^n$ with logarithmic poles, where $a>1$ is an irrational number. Assume that $c_o(\varphi_A)=c_o(\varphi_1)(=n-1+\frac{1}{a})$ ($c_o(\varphi_1)=n-1+\frac{1}{a}$ see [7]). Then $\nu(\varphi_A,o)<\nu(\varphi_1,o)(=1)$.*

Proof. As $\varphi_A\geq\varphi_1$, then it is clear that $\nu(\varphi_A,o)\leq\nu(\varphi_1,o)$.

We prove Lemma 2.3 by contradiction: if not, then $\nu(\varphi_A,o)=\nu(\varphi_1,o)(=1)$. By Lemma 2.2, it follows that $c_o(\varphi_A)\nu(\varphi_A,o)$ is a rational number, which contradicts $\nu(\varphi_A,o)c_o(\varphi_A)=1(n-1+\frac{1}{a})=n-1+\frac{1}{a}$. □

3. Proof of Theorem 1.2

We prove Theorem 1.2 by contradiction: if not, then there exists a plurisubharmonic function $\varphi_A\geq\varphi_1$ near o with logarithmic poles such that

$$c_o(\varphi_1)=c_o(\varphi_A) \tag{3.1}$$

($\varphi_A\geq\varphi_1\Rightarrow c_o(\varphi)\leq c_o(\varphi_A)$).

By inequalities 2.1 and 2.2 it follows that

$$\begin{aligned}c_o(\varphi_A)&\geq\frac{1}{e_1(\varphi_A)}+\cdots+\frac{e_{n-2}(\varphi_A)}{e_{n-1}(\varphi_A)}+\frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)}\\&\geq\frac{n-1}{e_{n-1}^{\frac{1}{n-1}}(\varphi_A)}+\frac{e_{n-1}(\varphi_A)}{e_n(\varphi_A)}\\&\geq\frac{n-1}{e_{n-1}^{\frac{1}{n-1}}(\varphi_A)}+\frac{e_{n-1}(\varphi_A)}{a}.\end{aligned} \tag{3.2}$$

Note that function $f(t):=\frac{n-1}{t^{\frac{1}{n-1}}}+\frac{t}{a}$ ($t\in(0,a^{\frac{n-1}{n}}]$) is strictly decreasing with respect to t. If $e_{n-1}(\varphi_A)\leq1$, then we have

$$\frac{n-1}{e_{n-1}^{\frac{1}{n-1}}(\varphi_A)}+\frac{e_{n-1}(\varphi_A)}{a}\geq n-1+\frac{1}{a}=c_o(\varphi), \tag{3.3}$$

moreover " $=$ " in inequality 3.3 holds if and only if $e_{n-1}(\varphi_A)=1$. If $e_{n-1}(\varphi_A)<1$, then it follows that $c(\varphi_A)>n-1+\frac{1}{a}$ (by inequality 3.2), which contradicts equality 3.1. Then it suffices to consider the case $e_{n-1}(\varphi_A)=1$.

Note that the second " $\geq$ " of inequality 3.2 is " $=$ " if and only if $e_1(\varphi_A)=\cdots=e_{n-1}(\varphi_A)=1$ (by $e_{n-1}(\varphi_A)=1$). By Lemma 2.3, it follows that $e_1(\varphi_A)<1$,

which implies that the second " $\geq$ " of inequality 3.2 is " $>$ ". Using inequality 3.3, we obtain that

$$c_o(\varphi_A) > n - 1 + \frac{1}{a},$$

which contradicts equality 3.1.

Then Theorem 1.2 has been proved.

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